Scaling of the Static Conductivity in the Quantum Hall Effect

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We performed a numerical study of the static diagonal conductivity σ_{xx} in the lowest Landau level for a disordered two-dimensional system in a magnetic field with short range impurity potentials. We find scaling of the conductivity peak at a single critical energy which is governed by both the localization length exponent $\nu = 2.37 \pm 0.05$ *and* an exponent $\eta' = 1.63 \pm 0.03$. We argue that η' can be identified with the fractal dimension $D(2)$. For the value of the critical conductivity we obtained $(0.5 \pm 0.02)e^2/h$ in agreement with the hypothesis of universality.

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The integer quantum Hall effect is closely related to a disorder driven localization-delocalization transition occuring in two-dimensional systems when the Fermi level lies at a critical energy E^c at the center of the Landau levels. The transition is characterized by a diverging localization length $\xi \propto (E-E^c)^{-\nu}$. In a previous experiment Koch [1] et al. were able to determine the critical exponent ν directly by studying the scaling of the peak width of the diagonal resistance ρ_{xx} as a function of the system size. The obtained value of $\nu = 2.3 \pm 0.1$ for the lowest three Landau levels agrees well with theoretical results. Extensive numerical approaches by Huckestein and Kramer [2] based on finite-size scaling of the localization length ξ yield $\nu = 2.34 \pm 0.04$ in the lowest Landau level. This is in remarkable agreement with the analytical value $\nu = \frac{7}{3}$ of Mil'nikov and Sokolov [3] or network models of quantum percolation [4,5] with $\nu = 2.43 \pm 0.18$. But the numerical fits rely on the assumption that there is a single point E^c at which ξ diverges. Huo and Bhatt [6] where able to rule out a scenario with two mobility edges and a finite region of extended states by identifying the first Chern character of the eigenstates, which is a measure of their extension and obtained $\nu = 2.4 \pm 0.1$. The critical conductivity at E^c is claimed to be universal with a value of $\frac{1}{2}$, irrespective of the range of the potential [4,5,7] and also within a semi-classical approximation [8]. Chalker and Daniell [9] have shown that the wave function under quantum Hall conditions shows anomalous diffusive behaviour at the transition point. In particular they find that the disorder-averaged density correlator $S(\boldsymbol{r}, \boldsymbol{r}'; E^c)$ scales with $|\boldsymbol{r} - \boldsymbol{r}'|^{-\eta}$. For electrons in the lowest Landau level, moving in a Gaussian white noise potential, they obtain $\eta = 0.38 \pm 0.04$. The exponent of anomalous diffusion η is related [10,11] to the generalized dimension $D(2)$ of the wave function by

$$\eta = 2 - D(2).$$

In the following we will present the results of a direct evaluation of the diagonal conductivity. We find scaling of the conductivity peak at a single critical energy which is governed by both critical exponents ν and η. Also an estimate for the critical conductivity is obtained which is compatible with the universal value.

We have developed a generalization of the MacKinnon recursion method [12] which allows to calculate the dynamical conductivity $\sigma_{xx}(\omega)$. The details of the method will be published elsewhere. As a first check of the method we used it to calculate the scaling properties of the static conductivity $\sigma_{xx}(0)$. We start from the Kubo formula expressed in terms of Green's functions $G(\zeta) = [\zeta - H]^{-1}$, $G = G(E + i\varepsilon)$ and $G_\omega = G(E + \hbar\omega + i\varepsilon)$. The real part of $\sigma_{xx}(\omega)$ at zero temperature is

$$\mathrm{Re}\,\sigma_{xx}(\omega, E_F) = \frac{e^2}{h}\frac{1}{v}\frac{1}{\hbar\omega}\int_{E_F-\hbar\omega}^{E_F} dE\, \mathrm{Re}\,\mathrm{Tr}\{(\hbar z)^2 x G_\omega x G^* - (\hbar\omega)^2 x G_\omega x G + 2i\varepsilon x^2 (G_\omega - G^*)\}.$$

E_F is the Fermi level, v denotes the volume of the system and $z = \omega + 2i\varepsilon$. The thermodynamic limit is achieved by increasing the system size $L \to \infty$ and finally decreasing $\varepsilon \to 0^+$, in order to retain all contributions from the spectrum of H. Thus we can proceed with a small, but fixed value of ε and study the dependence of $\sigma_{xx}^L(\omega, E_F)$ of the system size L. Following the course of MacKinnon we build the system recursively from a stack of slices and set up the corresponding system of iteration equations. The possibility of setting ε to zero when using appropriate boundary conditions has been discussed [12], but as we will show it is of advantage to retain control over the thermodynamic limiting process.

We study independent electrons of mass m and charge $q = -e$ moving in the x-y plane in a uniform perpendicular magnetic field B with $\boldsymbol{A} = (0, Bx, 0)$ and an impurity potential $V(\boldsymbol{r})$

$$\boldsymbol{H} = \frac{1}{2m}(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})^2 + V(\boldsymbol{r}).$$

We consider potentials with electron-hole symmetry and the correlation function

$$\overline{V(\boldsymbol{r} + \boldsymbol{b})V(\boldsymbol{r})} = \frac{u^2}{2\pi\lambda^2} e^{-b^2/2\lambda^2}.$$

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It is convenient to generate the potential matrix elements within the random matrix model instead of distributing individual random impurities, implying periodic boundary conditions across the slices. The random matrix model is based on the assumption that the scaling behaviour and all physically relevant features of the quantum Hall effect are independent of higher order correlation functions of the potential. So far the random matrix model has been successfully applied to the accurate calculation [2] of the critical exponent ν, the generalized fractal dimensions [13] and the equilibrium current distributions [14]. Even if $\lambda = 0$ a finite correlation length of the order of the magnetic length $l = (\hbar c/eB)^{1/2}$ is obtained [14].

We have studied systems with sizes $L = 10l$ to $50l$ for short ranged impurity potentials, $\lambda = 0$, and $\varepsilon = 5\times10^{-4}$ to $\varepsilon = 2\times10^{-3}$. The Fermi energy is varied in the range $-2 < E < 2$. The energy scale has been normalized to the second moment of the density of states $\rho(E)$. This is almost perfectly given by Wegner's function [15] as we have checked by an iterative calculation. In Fig. 1 the static conductivity $\sigma_{xx}^L(E)$ is shown as a function of the Fermi level E. More than 1.5×10^5 iterations for the two smaller systems and 10^5 iterations for the larger systems were performed. The statistical errors are less than half the symbol size. The conductivity peak becomes narrower with increasing system size and the peak height increases.

The peak width for each system size can be obtained with best accuracy by fitting a parabola $\sigma_{xx}^c - (E/\Delta E)^2$ within a small energy range $-1 < E < 1$ about the center (shaded region). In Fig. 2 the width parameter ΔE is plotted versus the system size. It can be seen that our data are compatible with one-parameter scaling for system widths larger than $L = 15$. The peak width is a statistically very well behaved quantity and we obtain an accurate estimate for the critical exponent $\nu = 2.37 \pm 0.05$ from a linear regression.

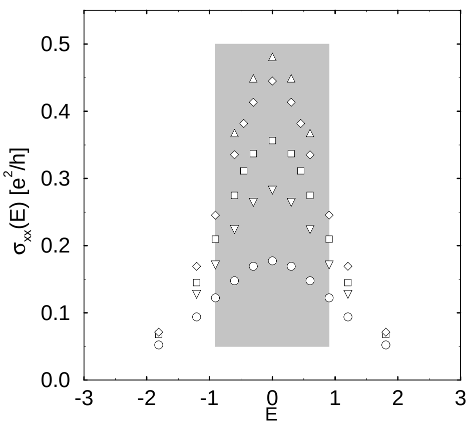

FIG. 1. The static conductivity $\sigma_{xx}^L(E)$ in units of e^2/h as a function of the Fermi level E. The system widths are $L = 10\,(\circ)$, $L = 15\,(\triangledown)$, $L = 20\,(\Box)$, $L = 30\,(\Diamond)$, $L = 40\,(\triangle)$ with $\varepsilon = 10^{-3}$. Data in the shaded region are used in calculating ΔE.

As shown in Fig. 1 the peak height increases with the system size. The claimed universal fix point value $\sigma_{xx}^c = e^2/2h$ is approached with a power law:

$$\Delta\sigma_{xx}^L = (\sigma_{xx}^c - \sigma_{xx}^L) \propto L^{-\eta'}.$$

In Fig. 3 the peak value of $\sigma_{xx}^L(E^c)$ is depicted for three values $\varepsilon = 2\times10^{-3}$, $\varepsilon = 1\times10^{-3}$ and $\varepsilon = 5\times10^{-4}$ and several system sizes $L = 10$ to 50.

Figure 4 shows the double logarithmic plot of the deviation $\Delta\sigma_{xx}^L$ from the critical conductivity $\sigma_{xx}^L(E^c,\varepsilon)$ for various system sizes and values of ε. Due to the finite value of ε the critical value is systematically shifted. We used the five largest systems for $\varepsilon = 2\times10^{-3}$, $\varepsilon = 1\times10^{-3}$ and the four largest systems for $\varepsilon = 5\times10^{-4}$ and adjusted η' simultaneously with $\sigma_{xx}(E^c,\varepsilon)$ for all values of ε. With $\varepsilon = 2\times10^{-3}$ we obtain a fixed point value of $\sigma_{xx}^c = 0.553$, with $\varepsilon = 1\times10^{-3}$ we obtain $\sigma_{xx}^c = 0.535$, and with $\varepsilon = 5\times10^{-4}$ we obtain $\sigma_{xx}^c = 0.516$ with the common exponent $\eta' = 1.63 \pm 0.03$. It can clearly be seen that the universal conductivity $\sigma_{xx}^c = e^2/2h$ is approached with decreasing ε and our estimate calculated for the smallest ε is very close to this value.

We observe that with decreasing ε the region in which scaling can be observed is shifted to increasing system sizes. The scaling relation of the static conductivity can be summarized in

$$\sigma_{xx}(E,L) = \sigma_{xx}^c - a_1\left(L^{\frac{1}{\nu}}(E-E^c)\right)^2 - a_2L^{-\eta'} + \cdots.$$

We will now establish a connection between the generalized dimension $D(2)$ of the wave function and the scaling index η'.

Wegner [15] defines an ensemble-averaged inverse participation ratio (IPR)

$$P^{(2)}(E) = \overline{\sum_{n,\boldsymbol{r}} |\Psi_n(\boldsymbol{r})|^4\delta(E-E_n)} \Big/ \overline{\sum_n \delta(E-E_n)}.$$

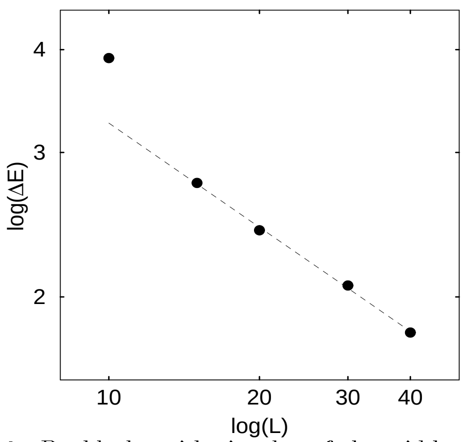

FIG. 2. Double logarithmic plot of the width ΔE of the $\sigma_{xx}^L(E)$ peak versus the system size L. The dashed line indicates the fitted exponent $\nu = 2.37 \pm 0.05$.

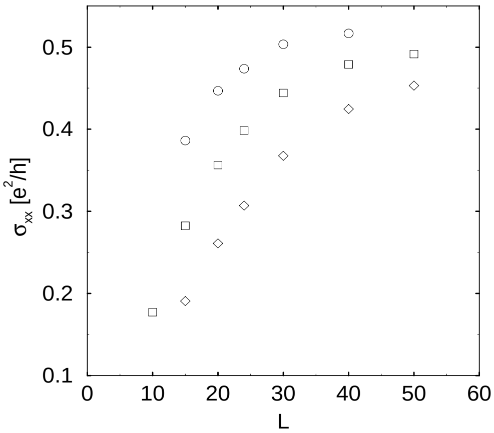

FIG. 3. The peak value of $\sigma^L_{xx}(E^c)$ for $\varepsilon = 2\times10^{-3}, (\circ)$, $\varepsilon = 1\times10^{-3}, (\square)$, $\varepsilon = 5\times10^{-4}, (\diamond)$ and systems of widths $L = 10$ to 50.

With the assumption that the average is invariant under translation $P^{(2)}$ can be expressed by the ensemble averaged two-point Green's function. With the density of states $\rho(E)$ and some fixed $\boldsymbol{r}_0$ it can be written as

$$P^{(2)}(E) = \lim_{\varepsilon\to0+} \frac{\varepsilon/\pi}{\rho(E)} \overline{|G(E+i\varepsilon, \boldsymbol{r}_0, \boldsymbol{r}_0)|^2},$$

showing that the IPR is proportional to the spectral component $p(E)$ of the return probability to $\boldsymbol{r}_0$. Wegner's critical exponent for $P^{(2)}(E) \propto (E-E^c)^{\pi(2)}$ is connected with the generalized fractal dimension by $\pi(2) = \nu D(2)$. From that we find

$$p(\xi) \propto \xi^{-D(2)}. \quad (1)$$

In the critical regime length scales and time scales are related by the dynamical exponent via $\xi^z = \tau$. Thus the return probability $p(\tau)$ at time τ scales like

$$p(\tau) = \tau^{-D(2)/z}.$$

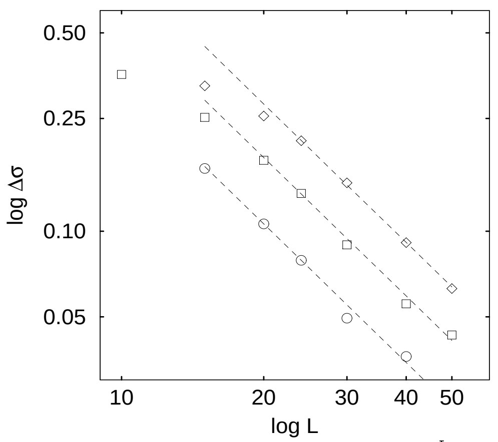

FIG. 4. The scaling of the deviation $\Delta\sigma^L_{xx}(E^c)$ from the critical conductivity σ^c_{xx} for $L = 10$ to 50. With $\varepsilon = 2\times10^{-3}, (\circ)$ we obtain $\sigma^c_{xx} = 0.553$, with $\varepsilon = 1\times10^{-3}, (\square)$ we obtain $\sigma^c_{xx} = 0.535$, for $\varepsilon = 5\times10^{-4}, (\diamond)$ we find $\sigma^c_{xx} = 0.516$, and $\eta' = 1.63\pm0.03$, indicated by dashed lines.

This relation has recently been studied [16] by computing the time evolution of wave packets built from states near the localization-delocalization transition. With $z = 2$ they obtained $D(2) = 1.62\pm0.04$.

The transport coefficients in the linear response can be connected with the scattering matrix formulation [17]. This is not surprising because the asymptotic behaviour of the Green's functions is closely related to the amplitudes in the scattering wave states. The reflection coefficient R is proportional to the modulus squared of the Green's function with both arguments $\boldsymbol{r}_0 \to \infty$ in the incoming channel. Therefore the scaling property of the averaged reflection coefficient R will be governed by the power law (1) of the return probability

$$R \propto \xi^{-D(2)}.$$

Expressing the conductivity by the Landauer-Büttiker formula for a two-probe measurement with ideal leads

$$\sigma = \frac{e^2}{h}T, \quad R = 1 - T,$$

where $T = \mathrm{Tr}\{t^+t\}$ and t is the matrix of transmission coefficients, we expect that the deviation of the conductivity from its fixed point value can be obtained from a finite size scaling analysis

$$\Delta\sigma^L = (\sigma^\infty - \sigma^L) \propto L^{-D(2)}.$$

Thus we estimate $\eta = 2 - D(2) = 2 - \eta' = 0.37\pm0.03$.

In conclusion we have shown that the scaling behaviour of the deviation of σ^L_{xx} from the universal conductivity σ^c_{xx} is governed by a power law with the critical exponent $\eta' = 1.63\pm0.03$. We argue that η' can be identified with the fractal dimension $D(2)$. Furthermore we find that σ^c_{xx} at the center of the lowest Landau level is equal to 0.50 ± 0.02 for short ranged potentials in excellent agreement with the hypothesis of universality. From the peak width of σ_{xx} we directly obtained the exponent $\nu = 2.37\pm0.05$. Our findings imply that η' could be measurable in low temperature experiments similar to that of determining the critical exponent ν.

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